On Lipschitz continuity of matrix functions

References:

Alexsandrov & Peller, "Estimates of operator moduli of continuity"

Bhatia, "Matrix Analysis"

→ Bhatia, "Modulus of continuity of the matrix absolute value"

Kato, "Continuity of the map $S \to |S|$ for linear operators"

McIntosh, "Counterexample to a question on commutators"

Q: Considering $A, B \in \mathbb{R}^{n \times n}$, Hermitian.

Consider $f: \mathbb{R} \to \mathbb{R}$ that's L-Lipschitz

$$\forall x, y, \quad |f(x) - f(y)| \le L|x-y|.$$

Let $f(A)$ denotes applying f to A's eigenvalues. $A = \sum \lambda_i u_i u_i^T$, $f(A) = \sum f(\lambda_i) u_i u_i^T$

Is it true that

$$\|f(A) - f(B)\|_* \le L \|A - B\|_* \ ?$$

Idea: Have a notion of "$Df(A)$", bound it.

$f(A) \longrightarrow f(B)$

$\le d(A, B)$.

Thm: Q is true for Frobenius norm.

Lem: If $\forall A$, $\forall X$ (not Hermitian),

$$\|f(A)X - Xf(A)\|_* \le C\|AX - XA\|_*,$$

then $\forall A, B$, $\forall X$

$$\|f(A)X - Xf(B)\|_* \le C\|AX - XB\|_*.$$

Pf: Consider some A, B, X.

Let $\hat{A} = \begin{bmatrix} A & \\ & B \end{bmatrix}$, $\hat{X} = \begin{bmatrix} & X \\ & \end{bmatrix}$ $\quad \square$

$f(\hat{A}) = \begin{bmatrix} f(A) & \\ & f(B) \end{bmatrix}$.

$$\|\underbrace{f(\hat{A})\hat{X} - \hat{X}f(\hat{A})}_{\begin{bmatrix} & f(A)X - Xf(B) \\ & \end{bmatrix}}\|_* \le C\|\underbrace{\hat{A}\hat{X} - \hat{X}\hat{A}}_{\begin{bmatrix} & AX - XB \\ & \end{bmatrix}}\|_*$$

Pf of Thm Suffices to bound

$$\|f(A)X - Xf(A)\|_F \le L\|AX - XA\|_F$$

WLOG A is diagonal, entries a_i.

$$\|f(A)X - Xf(A)\|_F^2 = \sum_{ij}\left[(f(a_i)-f(a_j))x_{ij}\right]^2 \le \sum_{i,j}\left[L(a_i-a_j)x_{ij}\right]^2$$

$$= L^2\|AX-XA\|_F^2 \quad \square$$

$$\begin{bmatrix} \text{—} f(a_1)X_1 \text{—} \\ \vdots \\ \text{—} f(a_n)X_n \text{—} \end{bmatrix} \qquad \begin{bmatrix} | & & | \\ f(a_1)X^{(1)} & & f(a_n)X^{(n)} \\ | & & | \end{bmatrix}$$

(Def $(C \circ X)_{ij} = c_{ij}x_{ij}$)

Idea Wanted to bound $\|C \circ X\|_*$; easy for Frobenius norm.

Time for spectral norm

$$\|f(A) - f(B)\| \lesssim (1+\log n)\, L\|A-B\|$$

$$\lesssim \left(1 + \log\frac{R}{\|A-B\|}\right) L\|A-B\| \leftarrow$$

R is range of eigvals of A, B $\lesssim \|A\| + \|B\|$

These are tight.

Ex. $f(x) = |x|$ $\qquad \||A| - |B|\| \lesssim \log(n)\|A-B\|$

$\Uparrow$ Lem

Thm $\||A|X - X|A|\| \lesssim \log(n)\|AX - XA\|$

Pf WLOG A is diagonal. $A = \begin{bmatrix} P & \\ & -Q \end{bmatrix}$ P, Q diagonal non-neg entries

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

$$AX - XA = \begin{bmatrix} PX_{11} - X_{11}P & PX_{12} + X_{12}Q \\ -QX_{21} - X_{21}P & -QX_{22} + X_{22}Q \end{bmatrix} \qquad A = \begin{bmatrix} P & \\ & -Q \end{bmatrix}$$

$$|A| = \begin{bmatrix} P & \\ & Q \end{bmatrix}$$

$$|A|X - X|A| = \begin{bmatrix} \underbrace{PX_{11} - X_{11}P}_{\|\cdot\| \leq \|AX-XA\|} & PX_{12} - X_{12}Q \\ QX_{21} - X_{21}P & \underbrace{QX_{22} - X_{22}Q}_{\|\cdot\| \leq \|AX-XA\|} \end{bmatrix}$$

Want to show $\|QX - XP\| \leq \log(n) \|QX + XP\|$.

then we're done.

$$\left(\begin{aligned} \| |A|X - X|A| \| &\leq \sum \|\text{blocks}\| \leq \sum \log n \,\|\text{blocks of } AX - XA\| \\ &\lesssim \log(n) \|AX - XA\|. \end{aligned} \right)$$

$$QX - XP = \underbrace{\left\{ \frac{q_i - p_j}{q_i + p_j} \right\}_{ij}}_{M} \circ (QX + XP)$$

Want $\|M\|_{SH} := \max\limits_{Z : \|Z\| = 1} \|M \circ Z\|$. $\xRightarrow{\text{reduces to bounding}}$ $\left\| \begin{bmatrix} \ddots & +1 \\ -1 & \ddots \end{bmatrix} \right\|_{SH}$