Problem Set 3: Polynomial Approximation

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Problem 1 (Polynomial approximation of monomials)**.** First, compute the Chebyshev coefficients of the monomial $m^{(n)}(x) = x^n$. (Doing this via $T_k(\frac{1}{2}(z + z^{-1})) = \frac{1}{2}(z^n + z^{-n})$ formulation may be easiest.) How small can k be such that the Chebyshev truncation $m_k^{(n)}$ a good approximation of $m^{(n)}$:

$$\|m^{(n)} - m_k^{(n)}\|_{[-1,1]} \le \varepsilon?$$

Solution. Substituting in $x = \frac{1}{2}(z + z^{-1})$, we get that

$$x^n = \frac{1}{2^n}(z + z^{-1})^n \tag{1}$$

$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} z^{k-(n-k)} \tag{2}$$

$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} z^{2k-n} \tag{3}$$

There's some annoyance involving parity. If n is odd, then

$$= \frac{1}{2^n}\Big(\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k} z^{2k-n} + \sum_{k=\lfloor n/2\rfloor+1}^{n}\binom{n}{k} z^{2k-n}\Big) \tag{4}$$

$$= \frac{1}{2^n}\sum_{k=\lfloor n/2\rfloor+1}^{n}\binom{n}{k} 2T_{2k-n}(x) \tag{5}$$

$$= \frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k} T_{n-2k}(x) \tag{6}$$

If n is even, then we get a constant term.

$$= \frac{1}{2^n}\Big(\binom{n}{n/2} + \sum_{k=0}^{n/2-1}\binom{n}{k} z^{2k-n} + \sum_{k=n/2+1}^{n}\binom{n}{k} z^{2k-n}\Big) \tag{7}$$

$$= \frac{1}{2^n}\Big(\binom{n}{n/2} + \sum_{k=n/2+1}^{n}\binom{n}{k} 2T_{2k-n}(x)\Big) \tag{8}$$

$$= \frac{1}{2^n}\Big(\binom{n}{n/2} + \sum_{k=0}^{n/2-1}\binom{n}{k} 2T_{n-2k}(x)\Big) \tag{9}$$

Roughly, the Chebyshev coefficient corresponding to a_ℓ is $2^{1-n}\binom{n}{(n-\ell)/2}$, up to parity issues. So, for the truncation $m_{2\ell}^{(n)}$, the tail bound is (again, morally),

$$m_{2\ell}^{(n)} = \sum_{k\geq \ell} \binom{n}{n/2-\ell} = \Pr[\mathrm{Bin}(n, 1/2) \leq n/2 - \ell]. \tag{10}$$

By a Chernoff bound, it suffices to choose $\ell = \mathcal{O}(\sqrt{n \log(1/\varepsilon)})$. See [SV14] for a more careful version of this argument. □

Problem 2 (Chebyshev interpolation [Tre19])**.** The *Chebyshev interpolant* of a function f, denoted p_d, is the unique degree-d polynomial such that $p_d(x_j) = f(x_j)$ for all $x_j = \cos(j\pi/d)$, $j = 0, 1, \ldots, d$. Prove that[1]

$$\|f(x) - p_d(x)\|_{[-1,1]} \leq 2 \sum_{\ell \geq d} |a_d|.$$

Hint: when is $T_k(x_j) = T_\ell(x_j)$ for all points $\{x_j\}$?

Solution. We will build the Chebyshev interpolant of the function and identify the maximal error associated with this interpolant.

First, a detour: observe that the following Chebyshev polynomials have the same value for $x = \frac{z+z^{-1}}{2}$ for $z^{2\nu n} = 1$ for any integer ν.

$$T_m, T_{2n-m}, T_{2n+m}, T_{4n-m}, T_{4n+m}, \ldots \tag{11}$$

This follows from the observation that $T_k(\frac{z+z^{-1}}{2}) = \frac{z^k+z^{-k}}{2}$ ([Tre19, Theorem 4.1]).

Now, consider the Chebyshev series associated with f:

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x) \tag{12}$$

Then, to produce an interpolant, we need to enforce the condition that $p_d(x_j) = f(x_j)$. This can be done by recognizing that $T_k(x), T_j(x)$ coincide for specific values of k, j depending on x. Then, at these values, you could rewrite the function as follows:

$$f(x_j) = \sum_{k=0}^{d} c_k \sum_{n \in S_k} T_n(x_j) \tag{13}$$

Where S_k are the set of Chebyshev polynomials taking the same value at x_j. We've already defined this set above, and can find an explicit form for c_k as follows ([Tre19, Theorem 4.2]):

$$c_0 = a_0 + a_{2n} + a_{4n} + \ldots \tag{14}$$
$$c_n = a_n + a_{3n} + \ldots \tag{15}$$
$$c_k = a_k + (a_{k+2n} + a_{-k+2n}) + (a_{k+4n} + a_{-k+4n}) + \ldots \tag{16}$$

[1] Recall that our approximation results used that $\|f(x) - f_d(x)\|_{[-1,1]} \leq \sum_{\ell \geq d} |a_d|$. So, Chebyshev interpolants p_d give the same results as Chebyshev truncations f_d, up to a constant factor. Interpolants have the advantage of being computable in $d+1$ function evaluations.

Therefore, the error in a dth degree truncation can be seen as follows:

$$f(x) - p_d(x) = \sum_{k=0}^{\infty} a_k T_k(x) - \sum_{k=0}^{d} c_k T_k(x) \tag{17}$$

$$= \sum_{k=d+1}^{\infty} a_k (T_k(x) - T_m(x)) \tag{18}$$

$$\leq \sum_{k=d+1}^{\infty} 2|a_k| \tag{19}$$

For $m(k, d)$. The second step follows because each of the terms between $0 \leq k \leq d$ cancel directly (each c_k contains an a_k within it), and the terms $k \geq d + 1$ occur because the coefficient of a_m within some c_k is still unmodified, just associated with a lower order Chebyshev polynomial $T_{m(k,d)}$ which coincides with T_k at the provided values of x_j. $\square$

Problem 3 (Jackson theorems, [Tre19]). Let $f : [-1, 1] \to \mathbb{R}$ be absolutely continuous and suppose f is of bounded variation, meaning that $\int_{-1}^{1} |f'(x)| \mathrm{d}x \leq V$. Then show that the Chebyshev coefficients of f satisfy

$$|a_k| \leq \frac{2V}{\pi k}.$$

Solution. See [Tre19, Theorem 7.1]; it's integration by parts on the integral equation for a_k. $\square$

Problem 4 (Optimal polynomial approximations; upper and lower bounds). Consider a function $f : [-1, 1] \to \mathbb{R}$ with a Chebyshev expansion $f(x) = \sum_{k \geq 0} a_k T_k(x)$. Prove that

$$\Big(\frac{1}{2} \sum_{k=n+1}^{\infty} a_k^2\Big)^{\frac{1}{2}} \leq \min_{\substack{p \in \mathbb{R}[x] \\ \deg p = n}} \|f(x) - p(x)\|_{[-1,1]} \leq \sum_{k=n+1}^{\infty} |a_k|$$

For what kind of Chebyshev coefficient decay is this characterization tight up to constants?

Solution. We follow [AA22, Proposition 2.2], but get an improved bound. The upper bound follows by taking $p(x) = f_n(x)$. The lower bound follows by bounding the max by the integral. Let $p(x) = \sum_{k=0}^{n} b_k T_k(x)$ be a degree-n polynomial. Take $b_k = 0$ for all $k > n$. Then

$$\begin{aligned}
\|f(x) - p(x)\|_{[-1,1]} &\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(\cos(\theta)) - p(\cos(\theta)))^2 \mathrm{d}\theta \\
&\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\sum_{k=0}^{\infty} (a_k - b_k) T_k(\cos(\theta))\Big)^2 \mathrm{d}\theta \\
&\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\sum_{k=0}^{\infty} (a_k - b_k) \cos(k\theta)\Big)^2 \mathrm{d}\theta
\end{aligned}$$

This is expression is the squared norm of the function $f(x) - p(x)$ under the inner product where $\cos(k\theta)$'s are orthogonal. So, this gives us the sum of squares of the coefficients.

$$
\begin{aligned}
&= \frac{1}{2\pi} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} (a_k - b_k)(a_\ell - b_\ell) \int_{-\pi}^{\pi} \cos(k\theta)\cos(\ell\theta) \mathrm{d}\theta \\
&= \frac{1}{2\pi} \sum_{k=0}^{\infty} (a_k - b_k)^2 \pi \\
&\geq \frac{1}{2} \sum_{k=n+1}^{\infty} (a_k - b_k)^2 \\
&= \frac{1}{2} \sum_{k=n+1}^{\infty} a_k^2.
\end{aligned}
$$

□

References

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[Tre19] Lloyd N. Trefethen. *Approximation theory and approximation practice, extended edition*. Extended edition [of 3012510]. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2019, pp. xi+363. ISBN: 978-1-611975-93-2. DOI: 10.1137/1.9781611975949 (pages 2, 3).