Query-optimal estimation of unitary channels in diamond distance

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Summary

Problem

Given an oracle to apply $Z \in \mathbb{U}(d)$, output a *classical description* of an estimate $U \in \mathbb{U}(d)$ such that

$$\mathrm{dist}_\diamond(Z, U) < \varepsilon$$

with probability $\geq \frac{2}{3}$.

Main result

$O(d^2/\varepsilon)$ queries to Z suffice.

The algorithm uses only *one* qudit.

$\Omega(d^2/\varepsilon)$ queries to Z, $Z^\dagger$, $\mathrm{c}Z$, or $\mathrm{c}Z^\dagger$ are necessary.

Diamond norm distance is equivalent up to constants to operator norm distance:

$$\mathrm{dist}_\diamond(U, V) \approx \mathrm{dist}(U, V) = \min_t \|U - e^{it}V\|_{\mathrm{op}}$$

Comparison to prior work

	# of queries	# of qudits	
[YRC20][1]	$d^{2.5}/\varepsilon$	$d^{2.5}/\varepsilon$	achieves optimal scaling in entanglement fidelity
process tomography	$\mathrm{poly}(d)/\varepsilon^2$	1	prepare-apply-measure
process tomography + algorithmic toolkit[2]	$d^2 \log(d)/\varepsilon$	$d \log(1/\varepsilon)$	requires cZ and $cZ^\dagger$
this work	d^2/ε	1	

[1] Yang, Renner, Chiribella. *Optimal universal programming of unitary gates*

[2] van Apeldoorn, Cornelissen, Gilyén, Nannicini. *Quantum tomography using state-preparation unitaries*

Outline

The algorithm

1. $O(d^2/\varepsilon^2)$ process tomography algorithm [standard]
2. $O(d^2/\varepsilon)$ "bootstrapping" algorithm from a $O(d^2/f(\varepsilon))$ "base" algorithm

The $O(d^2/\varepsilon^2)$ algorithm

Analyzing quantum process tomography

$O(d^2/\varepsilon^2)$ quantum process tomography

1. Pick a basis $|1\rangle, |2\rangle, \dots, |d\rangle$;
2. Prepare $O(d/\varepsilon^2)$ copies of $Z|k\rangle$ for every $k \in [d]$;
3. Run state tomography to get classical estimates $|u_k\rangle \in \mathbb{C}^d$ of $Z|k\rangle$;
4. Post-process to get some estimate U of Z.

State tomography guarantee

The output $|u_k\rangle$ satisfies that, for some $t_k \in [-\pi, \pi)$,

$$\| \underbrace{Z|k\rangle - e^{it_k}|u_k\rangle}_{\mathrm{err}_k} \| < \varepsilon.$$

The error err_k can be made Haar-random by "conjugating" by a Haar-random unitary, e.g. running $X^\dagger \mathcal{A}(X(Z|k\rangle))$.

Analyzing quantum process tomography

$$Z = \begin{pmatrix} | & | & \\ Z|1\rangle & Z|2\rangle & \cdots \\ | & | & \end{pmatrix} \approx \begin{pmatrix} | & | & \\ e^{it_1}|u_1\rangle & e^{it_2}|u_2\rangle & \cdots \\ | & | & \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} | & | & \\ |u_1\rangle & |u_2\rangle & \cdots \\ | & | & \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} e^{it_1} & & \\ & e^{it_2} & \\ & & \ddots \end{pmatrix}}_{\Phi} = U\Phi$$

$$\|Z - U\Phi\|_{\text{op}} = \Bigg\| \underbrace{\begin{pmatrix} | & | & \\ \text{err}_1 & \text{err}_2 & \cdots \\ | & | & \end{pmatrix}}_{\text{matrix of random columns with norm } \leq \varepsilon} \Bigg\|_{\text{op}} < 2\varepsilon$$

Finding relative phases

1. Run the procedure twice on Z and ZF (the discrete Fourier transform) to get U and V such that

$$\|Z - U\Phi\|_{\mathrm{op}} < \varepsilon \text{ and } \|ZF - V\Psi\|_{\mathrm{op}} < \varepsilon$$

   for unknown diagonal $\Phi, \Psi \in \mathbb{U}(d)$.

2. Compute $U^\dagger V$ to get a 2ε-estimate of

$$U^\dagger V \approx_{2\varepsilon} (Z\Phi^\dagger)^\dagger (ZF\Psi^\dagger) = \Phi F \Psi^\dagger = \begin{pmatrix} \phi_1\bar{\psi}_1 & \phi_1\bar{\psi}_2 & \phi_1\bar{\psi}_3 & \cdots \\ \phi_2\bar{\psi}_1 & \phi_2\omega_d\bar{\psi}_2 & \ddots & \\ \phi_3\bar{\psi}_1 & \ddots & & \\ \vdots & & & \end{pmatrix};$$

3. Read off the ϕ_i's from $U^\dagger V$ to get approximate phases $\widetilde{\Phi} \approx_{O(\varepsilon)} \Phi$.

The $O(d^2/\varepsilon)$ algorithm

Reducing error

Theorem

Consider a *base tomography* $\mathscr{A} : Z \mapsto U$ such that

$$\mathrm{dist}(Z, U) < c < 0.0001$$

using $O(Q)$ queries to Z. Then there is a *bootstrapped* $\overline{\mathscr{A}} : Z \mapsto U$ such that

$$\mathrm{dist}(Z, U) < \varepsilon$$

using $O(Q/\varepsilon)$ queries to Z.

Corollary

If $\mathscr{A}$ is the $O(d^2/\varepsilon^2)$-query 1-qudit algorithm,
then $\overline{\mathscr{A}}$ is a $O(d^2/\varepsilon)$-query 1-qudit algorithm.

A one-parameter warmup

We're told the unknown $Z \in \mathbb{U}(2)$ takes the form

$$Z = \begin{pmatrix} 1 & \\ & \phi \end{pmatrix}.$$

Then we can learn Z using a type of *phase estimation*.

1. Run the base $\mathscr{A}$ on Z^{2^k} for $k = 0, \ldots, \log_2 \frac{1}{\varepsilon}$ so that

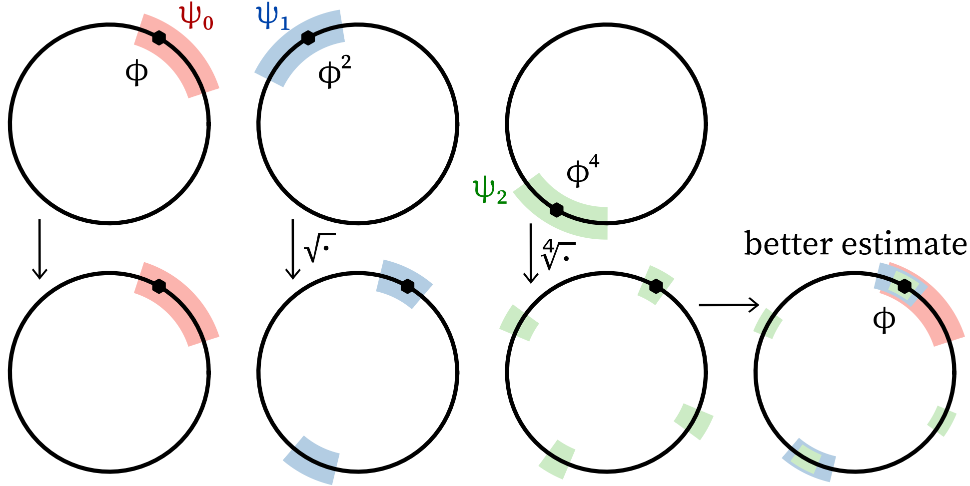
$$\mathrm{dist}(U_k, Z^{2^k}) < c$$

2. Extract the relative eigenvalues to get estimates ψ_k such that

$$|\psi_k - \phi^{2^k}| < c'$$

A one-parameter warmup

Weak estimates ψ of powers of ϕ



Preimages of the estimates

Extending the warmup to $\mathbb{U}(d)$

Algorithm idea

1. Run $\mathscr{A}$ on powers Z^{2^k}, up to $Z^{1/\varepsilon}$;
2. Receive the estimates $U_k \approx_c Z^{2^k}$;
3. Hope to compute $U \approx_\varepsilon Z$.

This fails: the hard case is when Z has eigenvalues of ± 1.

Error reduction near the identity

The -1 eigenvalue is the only hard case.

Let $U^{1/p}$ denote the "near-identity" root.

Lemma (taking roots improves error).

For $U, V \in \mathbb{U}(d)$ such that $\mathrm{dist}(U, I), \mathrm{dist}(V, I) \le 0.1$,

$$\mathrm{dist}(U^{1/p}, V^{1/p}) \le \frac{10}{p}\,\mathrm{dist}(U, V).$$

Proof idea.

Use that $\|X - Y\|_{\mathrm{op}}$ and $\|e^{iX} - e^{iY}\|_{\mathrm{op}}$ are equivalent for small Hermitian X, Y. □

The bootstrap

Let U_k be our current estimate to Z (where $U_0 = \mathrm{I}$).
Estimate the *remainder* $ZU_k^\dagger$ to stay close to I.

Algorithm

- For k from 0 to $T = \log_2(1/\varepsilon)$,
  1. Use $\mathscr{A}$ on $(ZU_k^\dagger)^{2^k}$ to get V_k
  2. Let $U_{k+1} = V_k^{1/2^k} U_k$
- Output U_{T+1}.

Query complexity

$O(Q)\sum_{k=0}^{T} 2^k = O(Q/\varepsilon)$

Space complexity

Same as base algorithm.

The bootstrap

Let U_k be our current estimate to Z (where $U_0 = \mathrm{I}$).
Estimate the *remainder* $ZU_k^\dagger$ to stay close to I.

Algorithm

- For k from 0 to $T = \log_2(1/\varepsilon)$, // induction: $U_k \approx Z$ with error $20c/2^k$
  1. Use $\mathscr{A}$ on $(ZU_k^\dagger)^{2^k}$ to get $V_k \approx (ZU_k^\dagger)^{2^k}$ with error c by $\mathscr{A}$ guarantee
     so $V_k^{1/2^k} \approx ZU_k^\dagger$ with error $10c/2^k$ by lemma
  2. Let $U_{k+1} = V_k^{1/2^k} U_k \approx Z$ with error $10c/2^k$
- Output U_{T+1}.

Discussion

Related work[3]

- We recover [YRC20]'s result for *entanglement fidelity* and *storage-and-retrieval*
- [HTFS22] gives a similar result for Hamiltonians

Open questions

- Can gate complexity be improved?
- Do these techniques extend to other problems?

[3] Yang, Renner, Chiribella. *Optimal universal programming of unitary gates*; Huang, Tong, Fang, Su. *Learning many-body Hamiltonians with Heisenberg-limited scaling*

Thank you!